QUALITATIVE ASSESSMENTS AND OPTIMAL DECISION UNDER UNCERTAINTY USING RECTANGULAR FUZZY NUMBERS

Abstract:

In crisis situations, uncertainty emerges as a distinct phenomenon, in which case we face both the ineffectiveness of most theoretical models of decision-making and the inaccuracy and incompleteness of available information.

It is therefore necessary to intensify theoretical efforts and to apply interdisciplinary approaches and transfers of methods in order to construct a set of procedures and techniques that should contribute to curbing this phenomenon in decision-making processes.

The transition from the state of crisis to the state of opportunity becomes the ground on which technocratic, structural, cultural and political approaches make a difference in managing.

The qualitative analysis of the level of uncertainty provides an optimal insight into the best management techniques and methods that may contribute to achieving development objectives under crisis conditions.

To conduct such an assessment process, we propose the use of rectangular fuzzy numbers. In this paper, first we develop a personal analysis of the notion of rectangular fuzzy numbers with an associated variable center of gravity and specific mathematical operations. Our research continues by the theoretical and applicable development of the "optimistic method (max-max)".

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Keywords: qualitative assessment, rectangular fuzzy numbers, "optimistic method (max-max)".

1. Introduction

The analysis of uncertainty in the business environment represents a constant priority for management. The current socio-economic context has prompted the development of research efforts in various fields on the dysfunctions generated by crises, with particular emphasis on their causes, emergence and manifestation and on their immediate and future consequences.

There is no "standard" method used as a panacea to reduce decision-making uncertainty. The technocratic approach on management is only valid on a free-competition market, with equal chances for everyone. In practice, this is a utopia. Knowledge-based economy changes the exogenous environment of decision, where the dominant aspects are, for example: data amount sometimes exceeds data-processing capacity, main competitors have the same information sources. A knowledge-based economy has also disadvantages, i.e. lower predictability.

The ongoing crisis is a particularly obvious symptom, which has brought about an open conflict between economic, political and social actors. It is therefore vital to properly recognise and assess uncertainties from the contingency perspective of the context of their occurrence, to comprehend their dynamic and the threats they pose to the history, culture and available resources of society, and to factor in the capacity for flexibility and adaptability, the technological potential, etc.

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The need for customised analysis based on the specifics of a particular field involves the identification and prediction of those situations or sets of specific particular circumstances, which directly influence activities as they unfold. They can be the result of actions and regulations at the micro and macroeconomic, social, and legal level, of technical and technological transformations, or of doctrines, theoretical and cultural concepts embraced at a particular moment and in a particular territorial space.

The interaction of these factors leads to the emergence of decision consequences that are difficult to anticipate and even harder to control, creating uncertainties that are difficult to absorb. The features which underlie the assessment of uncertainty differ from one field to another (i.e. economic, social, political, etc.), each featuring a panel of specific qualitative and quantitative indicators expressing the particular context and each resorting to different models of assessment and representation.

2. Qualitative assessment of the level of uncertainty using rectangular fuzzy numbers

In the most widely accepted points of view, decision-making under uncertainty conditions (e.g. I.Verboncu, O. Nicolescu, O. Snack, G. Aluja) are defined as situations in which the probability of achieving the objectives is high, yet there exist serious doubts as to the course of action to be taken. The emphasis is thus placed on factoring in a large number of variables, some of which are insufficiently controllable and whose evolution is difficult to forecast.

A different point of view from those outlined above deals with uncertainty as a key characteristic of life and as a source of progress, without distancing itself from the psychological view. Thus, in a progressive approach, Orio Giarini stated that “uncertainty may be described as the sum of all potential hazards around us, perceived or not”. Life’s uncertainties must be accepted as a reality, as the raw material for development and as an opportunity for maturity and real growth.

We propose the following as the main stages in the qualitative assessment of the level of uncertainty based on fuzzy numbers:

I. problem statement (analysis of the input data/information)
II. formulating the main lines of assessment
III. defining the groups of experts carrying out assessment in the various specialist fields
IV. qualitative assessment performed by experts (define the qualitative model; establish the qualitative assessment criteria for each specialist field; define indicators and scale items; complete the assessment scale – establish the actual measurement tool; uniform implementation of the assessment scales, assessment.
V. formalizing the process of fuzzy number-based strategy assessment (choose the best fuzzy-based method to formalise information; formalise information uncertainty – the absorption matrix; determine the logical assessment criteria)
VI. ranking the strategies (identify the optimal method of ranking the outcomes expressed as fuzzy numbers -regular ordering, distance from the optimum, etc.; rank the solutions.)
VII. determine the optimal option;
VIII. feedback provided to experts.

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There are many questions regarding the way in which an enterprise can become involved with logically grounded arguments in this management game\textsuperscript{6}.

Thus, the transition from the state of crisis to the state of opportunity becomes the ground on which technocratic, structural, cultural and political approaches make a difference in managing an organisation. In the knowledge management process, uncertainties emerge on the path from collecting data to gaining knowledge within the organisation and are dependent on technological constraints, accessibility, processing methods, theoretical models employed, and, last but not least, the human input\textsuperscript{7}.

3. Elementary rectangular fuzzy numbers with associated weight centers

A elementary rectangular fuzzy numbers\textsuperscript{8} is \( A=(a_m, a_M) \in \mathbb{I}_{dr} \) defined by \( \mu_A: \mathbb{R} \rightarrow [0,1] \),

\[
\mu_A = \begin{cases} 
1, & daca \ x \in [a_m, a_M] \\
0, & daca \ x \notin [a_m, a_M] 
\end{cases}, \quad \text{where} \ -\infty < a_m < a_M < +\infty
\]

We define the variables associated weight center: 
\[
G_\alpha(A) = a_\alpha + (\alpha - 1) \cdot S_\alpha(A) + \alpha \cdot S_\beta(A)
\]

where:
\[
\alpha \in \mathbb{R}, \alpha \in [0,1] - \text{uncertainty absorption indicator}
\]

Middle nucleus:
\[
a_\alpha(A) = \frac{a_m + a_M}{2} \in \mathbb{R}
\]

Left and right aria:
\[
S_\alpha(A) = a_\alpha(A) - a_m = \frac{a_M - a_m}{2} \in \mathbb{R} \quad S_\beta(A) = a_M - a_\alpha(A) = \frac{a_M - a_m}{2} \in \mathbb{R}
\]

Using two elementary rectangular fuzzy numbers with variables associated weight centers: \( A_\alpha = (a_\alpha; a_\alpha), B_\beta = (b_\beta; b_\beta) \in \mathbb{F}_{dr} \), and \( \alpha, \beta \in [0,1] \) the two levels of assumed uncertainty. We define these operations\textsuperscript{9}:

**Addition**: \( A_\alpha(+B_\beta) = C_\gamma = (c_1, c_2) \gamma = \begin{cases} 
muş1 = a_1 + b_1 \\
muş2 = a_2 + b_2 
\end{cases} \\
\gamma L_\mu(C) = \alpha L_\mu(A) + \beta L_\mu(B)
\]

**Substraction**: \( A_\alpha(-B_\beta) = C_\gamma = (c_1, c_2) \gamma = \begin{cases} 
muş1 = a_1 - b_2 \\
muş2 = a_2 - b_1 
\end{cases} \\
\gamma L_\mu(C) = \alpha L_\mu(A) + (1 - \beta) L_\mu(B)
\]

**Multiplication by a scalar** \( t \in \mathbb{R} \): \( C_\gamma = t A_\mu = \begin{cases} 
muş1 = (t * a_1, t * a_2), \gamma = \alpha & t \geq 0 \\
muş1 = (t * a_2, t * a_1), \gamma = 1 - \alpha & t < 0 
\end{cases} \\
\gamma = \frac{a_\mu * G(B) + G(A) * b_\mu}{2}
\]

**Multiplication**: \( A_\alpha * B_\beta = C_\gamma = (c_1, c_2) \gamma = \begin{cases} 
muş1 = a_1 * G(B) + G(A) * b_1 \\
muş2 = a_2 * G(B) + G(A) * b_2 
\end{cases} \\
\gamma = \alpha \beta
\]


\textsuperscript{7} C. Alecu , (2011), *Error analysis in management processes using fuzzy numbers*, Knowledge and Action within the Knowledge Based Society, Nr 11., Iasi, Editura Institutul European

\textsuperscript{8} Gherasim O. (2005), *Matematica numerelor fuzzy triunghiulare*, Iasi: Editura Performantica

mistic criterion ensures the selection of the variant with the highest yield

\[
\begin{align*}
c_1 &= \frac{a_1 \cdot G(B) + G(A) \cdot b_1}{2 \cdot G(B)^2} \\
c_2 &= \frac{a_2 \cdot G(B) + G(A) \cdot b_1}{2 \cdot G(B)^2} \\
y &= \alpha(1 - \beta)
\end{align*}
\]

**Division** \( A_y \cdot B_\beta = C_y = (c_1, c_2)_y \)

**Ranking criteria**

Ranking fuzzy numbers rectangular with associated weight centers shall be based on several successive criteria:

- **Weight ranking criteria:**
  \[
  \begin{align*}
  G(A_\alpha) > G(B_\beta) & \Rightarrow A_\alpha > B_\beta \\
  G(A_\alpha) < G(B_\beta) & \Rightarrow A_\alpha < B_\beta
  \end{align*}
  \]

- **Middle nucleus ranking criteria** (if \( G(A_\alpha) = G(B_\beta) \)):
  \[
  \begin{align*}
  N(A_\alpha) > N(B_\beta) & \Rightarrow A_\alpha > B_\beta \\
  N(A_\alpha) < N(B_\beta) & \Rightarrow A_\alpha < B_\beta
  \end{align*}
  \]

- **Ranking criteria based on nucleus length and the sign:**
  \[
  \begin{align*}
  G(A_\alpha) = G(B_\beta) & \Rightarrow (\text{sign}(A_\alpha)L_{sp}(A_\alpha) < \text{sign}(B_\beta)L_{sp}(B_\beta) \Rightarrow A_\alpha > B_\beta \\
  N(A_\alpha) = N(B_\beta) & \Rightarrow (\text{sign}(A_\alpha)L_{sp}(A_\alpha) > \text{sign}(B_\beta)L_{sp}(B_\beta) \Rightarrow A_\alpha < B_\beta
  \end{align*}
  \]

**The optimistic method (maxmax)**

The method starts from the idea that the optimum is given by the variant that has maximum advantages when the objective conditions have most favourable trends.

The principles of this method thus take the form of an **optimistic criterion** because it starts from the premise of a favourable, ascending evolution of the events (the hypothesis is risky and seldom occurs in practice). Therefore, rationally the optimum means the preference for the variant that will maximize the best possible results.

Be considered as optima that decision which satisfies the condition:

\[
D^* = \max_j \max_i \{I_{a_j} \cdot I_{D_i}\}
\]

where:

- \(I_{a_j}\) - consequences of decision taken by the manager for decision \(D_i\) evaluated by \(C_j\).
- \(I_{D_i}\) - elementary rectangular fuzzy numbers with associated weight centers
- \(D^*\) is the optim decision by optimistic criterion.

In the following case study we use a single indicator for absorbing uncertainty.

So the optimistic criterion ensures the selection of the variant with the highest yield potential but usually there are considerable risks when doing that.

The matrix of decision consequences obtained, the method needs the following steps to be taken: 1.- obtain a maximum for every line of the results that are the most certain and can be obtain in this way \(D^*_1 = \max_j \{R_{a_j} \cdot I_{D_i}\}\); 2.- obtain a maximum for these partial results:

\[
D^* = \max_j \{D^*_i\}\); 3.- identify the optimum variant (variant \(D_i\) corresponding to \(D^*\)).

**Notes:**

- the courses of action considered are feasible based on the available resources;
- the assessed characteristics are specific to the field of activity and structured according to the experts’ areas of competence;
- these characteristics are jointly determined by the groups of experts in order to enable a unified qualitative overview of courses of actions;
- a unified assessment system is defined, to ensure comparability of outcomes;
- qualitative assessment will yield a set of standardised values which will be applied to a unified system of decision-making techniques.
Qualitative assessments and optimal decision under uncertainty using rectangular fuzzy numbers

- A matrix of outcomes is derived; the elements of the matrix are rectangular fuzzy numbers.
- By mapping only the mode of defining the indicators of fuzzy numbers, a matrix of centres of gravity will result.

**Case study:**
Suppose a decision problem that has more valued course of actions (activities, relationships, products, services) through a Linkert scale by multiple knowledge workers. Linkert scale allows us to assess the quality characteristics and special assignment of numerical levels of preference.

We propose to issue a decision problem consists of three courses of action \((i = 1, 3)\) evaluated on the basis of four consequences/conditions of nature \((j = 1, 4)\).

Suppose that through the implemented management strategies a high degree of uncertainty absorption is obtained, so it satisfies the condition of assuming a single \(\alpha\).

\[\alpha_{ij} = \alpha, \quad \forall D_i \quad \text{or} \quad C_j, i = 1, 3 \quad \text{si} \quad j = 1, 4\]

We analyze the optimal solution for \(\alpha \in [0;1]\), as an expression of uncertainty absorbed.

For our problems, consequence matrix for each course of action will have the following form:

**Consequences matrix (single \(\alpha\))**

<table>
<thead>
<tr>
<th>(D_i / S_j)</th>
<th>(C_1)</th>
<th>(C_2)</th>
<th>(C_3)</th>
<th>(C_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_1)</td>
<td>([2;4]_\alpha)</td>
<td>([1;4]_\alpha)</td>
<td>([2;3]_\alpha)</td>
<td>([1;6]_\alpha)</td>
</tr>
<tr>
<td>(D_2)</td>
<td>([1;5]_\alpha)</td>
<td>([1;7]_\alpha)</td>
<td>([2;4]_\alpha)</td>
<td>([2;3]_\alpha)</td>
</tr>
<tr>
<td>(D_3)</td>
<td>([3;4]_\alpha)</td>
<td>([2;4]_\alpha)</td>
<td>([2;6]_\alpha)</td>
<td>([1;5]_\alpha)</td>
</tr>
</tbody>
</table>

1. To obtain a maximum for every line means to hierarchize the four rectangular fuzzy numbers. So the hierarchization criteria will be applied one by one if needed.

Thus we will first determine the centres of weight associated to every fuzzy number, and so a new matrix is obtained:

**Centres of weight associated matrix (single \(\alpha\))**

<table>
<thead>
<tr>
<th>(D_i / S_j)</th>
<th>(C_1)</th>
<th>(C_2)</th>
<th>(C_3)</th>
<th>(C_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_1)</td>
<td>(G_{11} = 2 + \alpha * (4-2))</td>
<td>(G_{12} = 1+ \alpha * (4-1))</td>
<td>(G_{13} = 2+ \alpha * (3-2))</td>
<td>(G_{14} = 1+ \alpha * (6-1))</td>
</tr>
<tr>
<td>(D_2)</td>
<td>(G_{21} = 1 + \alpha * (5-1))</td>
<td>(G_{22} = 1+ \alpha * (7-1))</td>
<td>(G_{23} = 2+ \alpha * (4-2))</td>
<td>(G_{24} = 2+ \alpha * (3-2))</td>
</tr>
<tr>
<td>(D_3)</td>
<td>(G_{31} = 3 + \alpha * (4-3))</td>
<td>(G_{32} = 2+ \alpha * (4-2))</td>
<td>(G_{33} = 2+ \alpha * (6-2))</td>
<td>(G_{34} = 1+ \alpha * (5-1))</td>
</tr>
</tbody>
</table>

This hierarchization, as seen in the matrix, depends on the variation of \(\alpha\) on the interval \([0;1]\) and therefore several inequations are to be solved:

\[
\begin{align*}
G_{11} &= 2 + \alpha * 2 \\
G_{12} &= 1+ \alpha * 3 \\
G_{13} &= 2 + \alpha \\
G_{14} &= 1+ \alpha * 5 \\
\end{align*}
\]

Depending on \(\alpha \in [0;1]\) we see that \(G_i^* = \max_j \{G_{ij}\}\) differs from case to case:

- If \(\alpha \in [0;\frac{1}{4}]\), then \(G_{12} < G_{14} < G_{13} < G_{11} \Rightarrow I_{12} < I_{14} < I_{13} < I_{11}\)
The additional application of the second hierarchization criteria is needed when the centres of weight are equal, for \( \alpha \in \left\{ \frac{1}{4}, \frac{1}{2}, \frac{1}{3} \right\} \).

- For \( \alpha = \frac{1}{4} \) (\( G_{13} = G_{14} \)), then will use second ranking criteria \( N(I_{13}) < N(I_{14}) \Rightarrow I_{13} < I_{14} \) - in this conditions: \( I_{12} < I_{13} < I_{14} < I_{11} \)

- For \( \alpha = \frac{1}{3} \) (\( G_{11} = G_{14} \)), we compare \( N(I_{13}) < N(I_{14}) \Rightarrow I_{13} < I_{14} \), then \( I_{12} < I_{13} < I_{11} < I_{14} \)

- For \( \alpha = \frac{1}{2} \) (\( G_{12} = G_{13} \)), we observe \( N(I_{12}) = N(I_{13}) \Rightarrow \) inconclusive – then will use third ranking criteria: \( \text{sign}(I_{12}) * L_{sp}(I_{12}) = 1^{*}3 \) \( \Rightarrow I_{12} < I_{13} \), and, in this conditions

\( I_{12} < I_{13} < I_{11} < I_{14} \)

- For \( \alpha = 1 \) (\( G_{11} = G_{12} \)), inconclusive, but \( N(I_{31}) > N(I_{34}) \Rightarrow I_{31} > I_{34} \) - then

\( I_{34} < I_{31} < I_{32} < I_{33} \)

After all
\[
\max\{I_{i,j}\} = I_{11}, \quad \alpha \in \left[ 0, \frac{1}{3} \right]
\]

On second line, the first criterion involves centers of gravity associated with the following hierarchy:

- \( \alpha \in \left[ 0, \frac{1}{5} \right] \) - \( G_{21} < G_{22} < G_{24} < G_{23} \)

- \( \alpha \in \left[ \frac{1}{5}, \frac{1}{4} \right] \) - \( G_{21} < G_{24} < G_{22} < G_{23} \)

- \( \alpha \in \left[ \frac{1}{4}, \frac{1}{3} \right] \) - \( G_{21} < G_{24} < G_{23} < G_{22} \)

- \( \alpha \in \left[ \frac{1}{3}, \frac{1}{2} \right] \) - \( G_{26} < G_{21} < G_{23} < G_{22} \)

- \( \alpha \in \left[ \frac{1}{2}, \frac{3}{2} \right] \) - \( G_{24} < G_{25} < G_{21} < G_{22} \)

The additional application of the second and third hierarchization criteria is needed when the centres of weight are equal, for \( \alpha \in \left\{ 0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{2} \right\} \).

After all
\[
\max\{I_{i,j}\} = I_{23}, \quad \text{for} \quad \alpha \in \left[ 0, \frac{1}{4} \right]
\]

On third line:

- \( \alpha \in \left[ 0, \frac{1}{3} \right] \) - \( I_{34} < I_{32} < I_{33} < I_{31} \)

- \( \alpha \in \left[ \frac{1}{3}, \frac{1}{2} \right] \) - \( I_{34} < I_{32} < I_{31} < I_{33} \)

- \( \alpha \in \left[ \frac{1}{2}, \frac{3}{2} \right] \) - \( I_{32} < I_{34} < I_{31} < I_{33} \)
Qualitative assessments and optimal decision under uncertainty using rectangular fuzzy numbers

1. We analyze the special situations for $\alpha \in \left[\frac{1}{3}; \frac{2}{3}; \frac{1}{3}\right]$. 

2. The second stage involves determining the optimum: $D^* = \max \{D_i^*\}$.

Determination of the optimum for single $\alpha$

<table>
<thead>
<tr>
<th>D_i /\alpha</th>
<th>Caz I</th>
<th>Caz II</th>
<th>Caz III</th>
<th>Caz IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha \in [0; \frac{1}{4}]$</td>
<td>$\alpha \in [\frac{1}{4}; \frac{1}{3}]$</td>
<td>$\alpha = \frac{1}{3}$</td>
<td>$\alpha \in [\frac{1}{3}; 1]$</td>
<td></td>
</tr>
<tr>
<td>D_1</td>
<td>$V_{1}^* = [2,4]$</td>
<td>$V_{1}^* = [2,4]$</td>
<td>$V_{1}^* = [1,6]$</td>
<td>$V_{1}^* = [1,6]$</td>
</tr>
<tr>
<td>D_2</td>
<td>$V_{2}^* = [2,4]$</td>
<td>$V_{2}^* = [1,7]$</td>
<td>$V_{2}^* = [1,7]$</td>
<td>$V_{2}^* = [1,7]$</td>
</tr>
<tr>
<td>D_3</td>
<td>$V_{3}^* = [3,4]$</td>
<td>$V_{3}^* = [3,4]$</td>
<td>$V_{3}^* = [3,4]$</td>
<td>$V_{3}^* = [2,6]$</td>
</tr>
</tbody>
</table>

**Caz I** $\alpha \in [0; \frac{1}{4}]$: in this case $\max(D_i^*) = D_1^*$. 

**Caz II** $\alpha \in [\frac{1}{4}; \frac{1}{3}]$: To obtain $\max(D_i^*)$ we rank the following centers of gravity:

$$
\begin{align*}
2 + \alpha; \\
1 + \alpha \cdot 7; \\
3 + \alpha
\end{align*}
$$

Then $\max(D_i^*) = D_1^*$. 

**Caz III** $\alpha = \frac{1}{3}$: in this case $\max(D_i^*) = D_1^*$. 

**Caz IV** $\alpha \in (\frac{1}{3}; 1)$: For different value of $\alpha$:

- $\alpha \in [\frac{1}{3}; \frac{2}{3}]$, then $\max(V_i^*) = V_1^*$
- $\alpha \in [\frac{2}{3}; 1]$, then $\max(V_i^*) = V_2^*$

3. The third stage involves identifying optimal variant

Depending on the evolution of the $\alpha$, the optimal variant would be:

- for $\alpha \in [0; \frac{2}{5}]$, $D^* = D_3$;
- for $\alpha \in [\frac{2}{5}; 1]$, $D^* = D_2$.

**Conclusions**

As we shall observe, multidisciplinarity is the hallmark of these management approach, as it integrates a series of sociological, psychological, statistical, mathematical, legal, etc. methods into an approach that is tailored to a particular activity.

As we can see, true this development of traditional methods with rectangular fuzzy numbers represents a technocratic answer to a call of interdisciplinary approach of decision-making under uncertainty. Through this demonstration, we see that it is possible to combine several different strategies and indicators, qualitative and quantitative.

By methods using rectangular fuzzy number with associated indicators to evaluate the quality of various relationships within and outside the organization the knowledge workers can generate solutions based on logical criteria in conditions of high uncertainty.
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